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# Conformally compactified Minkowski superspaces revisited

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## Abstract

Starting from the standard supertwistor realizations for conformally compactified  $\mathcal{N}$ -extended Minkowski superspaces in three and four space-time dimensions, we elaborate on alternative realizations in terms of graded two-forms on the dual supertwistor spaces. The construction is further generalized to the cases of 4D  $\mathcal{N} = 2$  and 3D  $\mathcal{N}$ -extended harmonic/projective superspaces. We present a superconformal Fourier expansion of tensor superfields on the 4D  $\mathcal{N} = 2$  harmonic/projective superspace.

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## 1 Introduction

Two years ago, Steven Weinberg [1] argued that the use of Dirac's projective light-cone realization for compactified four-dimensional (4D) Minkowski space [2] greatly

simplifies the calculation of Green's functions in conformally-invariant field theories (see also [3]). Clearly, this was not the first paper in which  $(d+2)$ -dimensional methods were used to study conformal theories in  $d$  dimensions, see, e.g., [4, 5, 6, 7, 8, 9, 10, 11]. Nevertheless, Weinberg's work has stimulated some renewed interest in Dirac's construction [2] and its implications. In particular, Ref. [12] considered a 4D  $\mathcal{N} = 1$  supersymmetric extension of the projective lightcone formalism of [2]. A generalization of [12] to the case  $\mathcal{N} > 1$  was attempted in [13]. Specifically, Refs. [12, 13] suggested to describe conformally compactified  $\mathcal{N}$ -extended Minkowski superspace,  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$ , in terms of graded two-forms on the dual supertwistor space. Here we demonstrate how to derive such a description starting from the standard supertwistor realization for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$ , see [14] and references therein. Our discussion is more complete and differs in some details.

We also present a new realization for compactified 4D  $\mathcal{N} = 2$  harmonic/projective superspace building on the formulation given in [14]. Finally, we generalize our construction to the case of conformally compactified 3D  $\mathcal{N}$ -extended Minkowski superspace  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  and its harmonic/projective extensions [15] (see [16] for an alternative construction).

This paper is organized as follows. Section 2 is devoted to non-supersymmetric warm-up exercises. Here we describe three different realizations for conformally compactified Minkowski space in four dimensions,  $\overline{\mathbb{M}}^4$ , and prove their equivalence. In section 3 we start by recalling the standard supertwistor realization for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$ . Then we introduce a novel bi-supertwistor realization for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$ . After that we prove the equivalence of these two realizations. Section 4 is devoted to a new realization for compactified 4D  $\mathcal{N} = 2$  harmonic/projective superspace,  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$ . In sections 5 to 7, we generalize the construction to three space-time dimensions. Concluding comments are given in section 8. The main body of the paper is accompanied by a technical appendix devoted to spinors in  $4 + 2$  dimensions.

## 2 Compactified 4D Minkowski space

In this section we describe three different realizations for conformally compactified Minkowski space in four dimensions.

## 2.1 Dirac's realization in $d$ space-time dimensions

We start by recalling the projective light cone formalism of [2] (see also Weyl's book [17]). Consider a flat space  $\mathbb{R}^{d,2}$  parametrized by Cartesian coordinates

$$X^{\hat{a}} = (X^a, X^{d+1}, X^{d+2}) , \quad a = 0, 1, \dots, d-1 \quad (2.1)$$

and endowed with the metric  $\eta_{\hat{a}\hat{b}} = \text{diag}(-1, +1, \dots, +1, -1)$ . Let us introduce the cone  $\mathcal{C}$  in  $\mathbb{R}^{d,2}$  defined by

$$\eta_{\hat{a}\hat{b}} X^{\hat{a}} X^{\hat{b}} = 0 . \quad (2.2)$$

The space of all straight lines belonging to  $\mathcal{C}$  and passing through the origin of  $\mathbb{R}^{d,2}$  is known as Dirac's conformal space [2] or compactified Minkowski space,  $\overline{\mathbb{M}}^d$ . It can be defined as the quotient space of  $\mathcal{C} - \{0\}$  with respect to the equivalence relation

$$X^{\hat{a}} \sim \lambda X^{\hat{a}} , \quad \lambda \in \mathbb{R} - \{0\} \quad (2.3)$$

which identifies all points on a straight line in  $\mathbb{R}^{d,2}$ . The conformal group in  $d$  dimensions,  $\text{O}(d, 2)/\mathbb{Z}_2$ , with  $\mathbb{Z}_2 = \{\pm \mathbb{1}_{d+2}\}$ , naturally acts on  $\overline{\mathbb{M}}^d$ . It may be seen that  $\overline{\mathbb{M}}^d$  is a homogeneous space of the connected conformal group, which is  $\text{SO}_0(d, 2)$  if  $d$  is odd and  $\text{SO}_0(d, 2)/\mathbb{Z}_2$  if  $d$  is even.

It follows that the global structure of  $\overline{\mathbb{M}}^d$  as a topological space is

$$\overline{\mathbb{M}}^d = (S^{d-1} \times S^1)/\mathbb{Z}_2 . \quad (2.4)$$

Indeed, the constraint (2.2) and the 'gauge' freedom (2.3) can be used to choose  $X^{\hat{a}}$  such that

$$(X^0)^2 + (X^{d+2})^2 = \sum_{i=1}^{d-1} (X^i)^2 + (X^{d+1})^2 = 1 . \quad (2.5)$$

For such a choice, the equivalence relation (2.3) still allows us to identify  $X^{\hat{a}}$  and  $-X^{\hat{a}}$ , which is the reason for  $\mathbb{Z}_2$  in (2.4). In four dimensions,  $\overline{\mathbb{M}}^4$  is the same topological space as the group manifold  $\text{U}(2)$ . In three dimensions,  $\overline{\mathbb{M}}^3$  can be identified with  $\text{U}(2)/\text{O}(2)$ .

Minkowski space  $\mathbb{M}^d \equiv \mathbb{R}^{d-1,1}$  can be identified, e.g., with the open *dense* domain of  $\overline{\mathbb{M}}^d$  on which  $X^{d+1} + X^{d+2} \neq 0$ .<sup>1</sup> This domain can be parametrized by variables

$$x^a = \frac{X^a}{X^{d+1} + X^{d+2}} \quad (2.6)$$

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<sup>1</sup>The closed subset  $C_3 := \overline{\mathbb{M}}^d - \mathbb{M}^d$  can be identified with a lightcone in  $\mathbb{R}^{d-1,1}$ . Indeed, the points of  $C_3$  can be uniquely parametrized by  $(D+2)$ -vectors of the form  $X^{\hat{a}} = (z^a, -\frac{1}{2}, \frac{1}{2})$ , where  $z^a \in \mathbb{R}^{d-1,1}$  is null,  $z^a z_a = 0$ .

which are invariant under the identification (2.3). In terms of these coordinates, one obtains a standard action of the conformal group in  $\mathbb{M}^d$ . Thus  $x^a$  can be identified with Cartesian coordinates for Minkowski space.

In the remainder of this section our consideration is restricted to the case  $d = 4$ .

## 2.2 Twistor realization

Here we recall the so-called twistor realization<sup>2</sup> of compactified Minkowski space  $\overline{\mathbb{M}}^4$  as the set of null two-dimensional subspaces in the twistor space,  $\mathbb{C}^4$ , equipped with the inner product

$$\langle T, S \rangle = T^\dagger \Omega S, \quad \Omega = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad (2.7)$$

for any twistors  $T, S \in \mathbb{C}^4$ . The components of a twistor  $T$  and its dual  $\bar{T} := T^\dagger \Omega$  are denoted as

$$T = (T_{\hat{\alpha}}) = \begin{pmatrix} f_{\alpha} \\ \bar{h}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{T} = (\bar{T}^{\hat{\alpha}}) = (h^{\alpha}, \bar{f}_{\dot{\alpha}}). \quad (2.8)$$

By construction, the inner product (2.7) is invariant under the action of the group  $\text{SU}(2, 2)$  which is the two to one covering of the group  $\text{SO}_0(4, 2)$ , which in turn is the two to one covering of the connected conformal group,  $\text{SO}_0(4, 2)/\mathbb{Z}_2$ . The elements of  $\text{SU}(2, 2)$  will be represented by block matrices

$$g = (g_{\hat{\alpha}}^{\hat{\beta}}) = \begin{pmatrix} A & \text{i}B \\ -\text{i}C & D \end{pmatrix} \in \text{SL}(4, \mathbb{C}), \quad g^\dagger \Omega g = \Omega, \quad (2.9)$$

where  $A, B, C$  and  $D$  are  $2 \times 2$  matrices. Under the action of  $\text{SU}(2, 2)$ , the twistor  $T_{\hat{\alpha}}$  and its dual  $\bar{T}^{\hat{\alpha}}$  transform as follows:

$$T_{\hat{\alpha}} \rightarrow g_{\hat{\alpha}}^{\hat{\beta}} T_{\hat{\beta}}, \quad \bar{T}^{\hat{\alpha}} \rightarrow \bar{T}^{\hat{\beta}} (g^{-1})_{\hat{\beta}}^{\hat{\alpha}}. \quad (2.10)$$

These two representations are inequivalent.

We denote by  $\overline{\mathbb{M}}_{\text{T}}^4$  the space of null two-planes through the origin in  $\mathbb{C}^4$ . Given such a two-plane, it is generated by two linearly independent twistors,  $T^\mu = (T^1, T^2)$ , subject to the null conditions

$$\langle T^\mu, T^\nu \rangle \equiv \bar{T}^\mu T^\nu = 0, \quad \mu, \nu = 1, 2. \quad (2.11)$$

---

<sup>2</sup>This realization has been known since the 1960s, see [18, 19, 20, 9, 21] and references therein.

The basis chosen,  $\{T^\mu\}$ , is defined only modulo the equivalence relation

$$\{T^\mu\} \sim \{\tilde{T}^\mu\} , \quad \tilde{T}^\mu = T^\nu C_\nu^\mu , \quad C \in \text{GL}(2, \mathbb{C}) . \quad (2.12)$$

The two bases,  $\{T^\mu\}$  and  $\{\tilde{T}^\mu\}$ , define one and the same two-plane in  $\mathbb{C}^4$ .

Minkowski space,  $\mathbb{M}^4 \equiv \mathbb{R}^{3,1}$ , can be embedded into  $\overline{\mathbb{M}}_{\text{T}}^4$  as a dense open subset. Consider the open domain of  $\overline{\mathbb{M}}_{\text{T}}^4$  consisting of those null two-planes which have the form

$$(T_{\hat{\alpha}}{}^\mu) = \begin{pmatrix} F_{\alpha}{}^\mu \\ H^{\dot{\alpha}\mu} \end{pmatrix} , \quad \det(F_{\alpha}{}^\mu) \neq 0 . \quad (2.13)$$

Then, choosing  $C = F^{-1}$  in (2.12) and making use of the null condition (2.11) leads to the basis

$$(T_{\hat{\alpha}}{}^\mu) = \begin{pmatrix} \delta_{\alpha}{}^{\beta} \\ -i x^{\dot{\alpha}\beta} \end{pmatrix} , \quad x^{\dot{\alpha}\beta} := x^m (\tilde{\sigma}_m)^{\dot{\alpha}\beta} , \quad x^m = (x^m)^* . \quad (2.14)$$

The group element (2.9) acts on  $\tilde{x} = (x^{\dot{\alpha}\beta})$  by the fractional linear transformation

$$\tilde{x}' = (C + D\tilde{x})(A + B\tilde{x})^{-1} , \quad (2.15)$$

which is a standard conformal transformation in Minkowski space  $\mathbb{M}^4$ . Therefore the open domain of  $\overline{\mathbb{M}}_{\text{T}}^4$  introduced can indeed be identified with Minkowski space.

Let us show that  $\overline{\mathbb{M}}_{\text{T}}^4$  is equivalent to the compactified Minkowski space introduced in the previous subsection. Using the two twistors  $T^\mu$ , which describe a point in  $\overline{\mathbb{M}}_{\text{T}}^4$ , we define the following  $4 \times 4$  matrix:

$$Y_{\hat{\alpha}\hat{\beta}} := T_{\hat{\alpha}}{}^\mu T_{\hat{\beta}}{}^\nu \varepsilon_{\mu\nu} , \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} , \quad \varepsilon_{12} = -1 . \quad (2.16)$$

This matrix is antisymmetric,

$$Y_{\hat{\alpha}\hat{\beta}} = -Y_{\hat{\beta}\hat{\alpha}} , \quad (2.17)$$

and defined modulo the equivalence relation

$$Y_{\hat{\alpha}\hat{\beta}} \sim c Y_{\hat{\alpha}\hat{\beta}} , \quad c = \det(C_{\hat{\alpha}}{}^{\hat{\beta}}) \in \mathbb{C} - \{0\} . \quad (2.18)$$

It is characterized by the algebraic properties

$$Y_{[\hat{\alpha}\hat{\beta}} Y_{\hat{\gamma}]\hat{\delta}} = 0 \quad \longleftrightarrow \quad Y_{[\hat{\alpha}\hat{\beta}} Y_{\hat{\gamma}\hat{\delta}]} = 0 . \quad (2.19)$$

Using the dual twistors  $\bar{T}$ , we can define

$$\bar{Y}^{\hat{\alpha}\hat{\beta}} := \varepsilon_{\mu\nu} \bar{T}^{\mu\hat{\alpha}} \bar{T}^{\nu\hat{\beta}} . \quad (2.20)$$

The matrices  $Y = (Y_{\hat{\alpha}\hat{\beta}})$  and  $\tilde{Y} = (\bar{Y}^{\hat{\alpha}\hat{\beta}})$  are related to each other as follows

$$\tilde{Y} = -\Omega Y^\dagger \Omega . \quad (2.21)$$

As a consequence of (2.11), we have

$$\bar{Y}^{\hat{\alpha}\hat{\gamma}} Y_{\hat{\gamma}\hat{\beta}} = 0 . \quad (2.22)$$

Introduce six-vectors  $Y^{\hat{a}}$  and  $\bar{Y}^{\hat{a}}$  defined by

$$Y^{\hat{a}} := \frac{1}{4}(\Sigma^{\hat{a}})_{\hat{\beta}\hat{\gamma}} Y^{\hat{\beta}\hat{\gamma}} , \quad \bar{Y}^{\hat{a}} := \frac{1}{4}(\Sigma^{\hat{a}})_{\hat{\beta}\hat{\gamma}} \bar{Y}^{\hat{\beta}\hat{\gamma}} , \quad (2.23)$$

with  $Y^{\hat{\alpha}\hat{\beta}} := \frac{1}{2}\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} Y_{\hat{\gamma}\hat{\delta}}$ . These six-vectors are mutually null,

$$\eta_{\hat{a}\hat{b}} Y^{\hat{a}} Y^{\hat{b}} = \eta_{\hat{a}\hat{b}} \bar{Y}^{\hat{a}} \bar{Y}^{\hat{b}} = \eta_{\hat{a}\hat{b}} Y^{\hat{a}} \bar{Y}^{\hat{b}} = 0 . \quad (2.24)$$

The first two relations follow from (A.36), while the third is a consequence of (2.22). Moreover, it follows from (2.22) that

$$Y^{\hat{a}} \bar{Y}^{\hat{b}} (\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}}^{\hat{\delta}} = 0 , \quad (2.25)$$

and therefore the mutually conjugate six-vectors  $Y^{\hat{a}}$  and  $\bar{Y}^{\hat{a}} = (Y^{\hat{a}})^*$  are linearly dependent. This means that

$$Y^{\hat{a}} = e^{i\varphi} X^{\hat{a}} , \quad \bar{Y}^{\hat{a}} = e^{-i\varphi} X^{\hat{a}} , \quad \varphi \in \mathbb{R} , \quad (2.26)$$

for some real null six-vector  $X^{\hat{a}}$ . By construction,  $X^{\hat{a}}$  is defined modulo the equivalence relation (2.3). In summary, we have defined an injective<sup>3</sup> map  $F : \overline{\mathbb{M}}_{\text{T}}^4 \rightarrow \overline{\mathbb{M}}^4$ . This map is in fact onto, and therefore one-to-one. To prove this, we associate with  $X^{\hat{a}}$  the antisymmetric matrix

$$X_{\hat{\alpha}\hat{\beta}} := X^{\hat{a}} (\Sigma_{\hat{a}})_{\hat{\beta}\hat{\gamma}} . \quad (2.27)$$

For its conjugate  $\tilde{X} = (\bar{X}^{\hat{\alpha}\hat{\beta}})$  defined according to (2.21) we obtain

$$\bar{X}^{\hat{\alpha}\hat{\beta}} = \frac{1}{2}\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} X_{\hat{\gamma}\hat{\delta}} \equiv X^{\hat{\alpha}\hat{\beta}} . \quad (2.28)$$

The matrices  $X_{\hat{\alpha}\hat{\beta}}$  and  $\bar{X}^{\hat{\alpha}\hat{\beta}}$  obey the properties (2.19) and (2.22). It turns out that  $X_{\hat{\alpha}\hat{\beta}}$  defines a null two-plane in  $\mathbb{C}^4$ . This follows from the discussion below.

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<sup>3</sup>For completeness, we prove that the map  $F$  is injective. Suppose this is not true and there exist two different two-planes that are mapped by  $D$  to the same point of  $\overline{\mathbb{M}}^4$ . Then we can choose the bases  $T^\mu = (T, S)$  and  $T'^\mu = (T', S')$  for the two-planes under consideration such that  $T \wedge S = T' \wedge S'$  (here we think of  $Y_{\hat{\alpha}\hat{\beta}}$  defined by (2.16) as an element of  $\wedge^2 \mathbb{C}^4$ ). Let us introduce a basis for  $\mathbb{C}^4$  consisting of  $T^\mu$  and two additional vectors  $W^\mu$ . Without loss of generality, we can choose  $T'$  and  $S'$  such that  $T' = T + t_\mu W^\mu$  and  $S' = S + s_\mu W^\mu$ , for some complex parameters  $t_\mu$  and  $s_\mu$ . Now, since  $T^\mu$  and  $W^\mu$  are linearly independent, the condition  $T \wedge S = T' \wedge S'$  tells us that  $t_\mu = s_\mu = 0$ . As a result,  $T^\mu$  and  $T'^\mu$  define one and the same two-plane, which contradicts the assumption.

### 2.3 Bi-twistor realization

There exists an alternative realization for  $\overline{\mathbb{M}}_{\text{T}}^4$ . Let us denote by  $\mathcal{L}$  the set of all *non-zero* complex antisymmetric matrices  $Y_{\hat{\alpha}\hat{\beta}} = -Y_{\hat{\beta}\hat{\alpha}}$  which obey the algebraic constraints

$$Y_{[\hat{\alpha}\hat{\beta}}Y_{\hat{\gamma}\hat{\delta}]} = 0 , \quad (2.29a)$$

$$\bar{Y}^{\hat{\alpha}\hat{\gamma}}Y_{\hat{\gamma}\hat{\beta}} = 0 , \quad \tilde{Y} := -\Omega Y^\dagger \Omega . \quad (2.29b)$$

We introduce the quotient space  $\overline{\mathbb{M}}_{\text{BT}}^4 = \mathcal{L}/\sim$ , where the equivalence relation is defined by

$$Y_{\hat{\alpha}\hat{\beta}} \sim c Y_{\hat{\alpha}\hat{\beta}} , \quad c \in \mathbb{C} - \{0\} . \quad (2.30)$$

We will show that  $\overline{\mathbb{M}}_{\text{BT}}^4$  can be naturally identified with  $\overline{\mathbb{M}}_{\text{T}}^4$ .

It follows from (2.29a) that  $Y_{\hat{\alpha}\hat{\beta}}$  is decomposable, that is

$$Y_{\hat{\alpha}\hat{\beta}} = T_{\hat{\alpha}}S_{\hat{\beta}} - T_{\hat{\beta}}S_{\hat{\alpha}} , \quad (2.31)$$

for some linearly independent twistors  $T_{\hat{\alpha}}$  and  $S_{\hat{\alpha}}$ , see e.g. [21] for the proof. Now, eq. (2.29b) is equivalent to

$$\bar{Y}^{\hat{\alpha}\hat{\gamma}}Y_{\hat{\gamma}\hat{\beta}} = \bar{T}^{\hat{\alpha}}\left\{\langle S, T \rangle S_{\hat{\beta}} - \langle S, S \rangle T_{\hat{\beta}}\right\} + \bar{S}^{\hat{\alpha}}\left\{\langle T, S \rangle T_{\hat{\beta}} - \langle T, T \rangle S_{\hat{\beta}}\right\} = 0 . \quad (2.32)$$

Since  $\bar{T}^{\hat{\alpha}}$  and  $\bar{S}^{\hat{\alpha}}$  are linearly independent dual twistors, the expressions in figure brackets must vanish. Since  $T_{\hat{\alpha}}$  and  $S_{\hat{\alpha}}$  are linearly independent, we conclude that

$$\langle T, T \rangle = \langle S, T \rangle = \langle T, S \rangle = \langle S, S \rangle = 0 , \quad (2.33)$$

and therefore the two-plane in  $\mathbb{C}^4$  associated with  $T_{\hat{\alpha}}$  and  $S_{\hat{\alpha}}$  is null. We finally choose  $T^\mu = (T, S)$ .

## 3 Compactified 4D Minkowski superspace

Supertwistor space  $\mathbb{C}^{4|\mathcal{N}}$  was introduced by Ferber [22] as a supersymmetric extension of the twistor space. The elements of  $\mathbb{C}^{4|\mathcal{N}}$  are called supertwistors. We use capital boldface letters,  $\mathbf{T}, \mathbf{S}, \dots$ , to denote supertwistors, for instance

$$\mathbf{T} = (\mathbf{T}_A) = \begin{pmatrix} \mathbf{T}_{\hat{\alpha}} \\ \mathbf{T}_i \end{pmatrix} , \quad i = 1, \dots, \mathcal{N} . \quad (3.1)$$



The supertwistor space is equipped with the inner product

$$\langle \mathbf{T}, \mathbf{S} \rangle = \mathbf{T}^\dagger \boldsymbol{\Omega} \mathbf{S}, \quad \boldsymbol{\Omega} = \begin{pmatrix} 0 & \mathbb{1}_2 & 0 \\ \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & -\mathbb{1}_{\mathcal{N}} \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\ 0 & -\mathbb{1}_{\mathcal{N}} \end{pmatrix}. \quad (3.2)$$

This inner product is invariant under the  $\mathcal{N}$ -extended superconformal group  $\mathrm{SU}(2, 2|\mathcal{N})$  spanned by supermatrices of the form

$$g = (g_A{}^B) \in \mathrm{SL}(4|\mathcal{N}), \quad g^\dagger \boldsymbol{\Omega} g = \boldsymbol{\Omega}. \quad (3.3)$$

Associated with a supertwistor  $\mathbf{T}$ , eq. (3.1), is its dual

$$\bar{\mathbf{T}} := \mathbf{T}^\dagger \boldsymbol{\Omega} = (\bar{\mathbf{T}}^A) = (\bar{\mathbf{T}}^{\hat{\alpha}}, -\bar{\mathbf{T}}^i), \quad \bar{\mathbf{T}}^i := (\mathbf{T}_i)^*. \quad (3.4)$$

The superconformal group acts on  $\mathbf{T}_A$  and  $\bar{\mathbf{T}}^A$  as follows:

$$\mathbf{T}_A \rightarrow g_A{}^B \mathbf{T}_B, \quad \bar{\mathbf{T}}^A \rightarrow \bar{\mathbf{T}}^B (g^{-1})_B{}^A. \quad (3.5)$$

### 3.1 Supertwistor realization

In complete analogy with the bosonic construction described in the previous section, compactified Minkowski superspace<sup>4</sup>  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  is defined to be the space of null two-planes through the origin in  $\mathbb{C}^{4|\mathcal{N}}$  [23]. Given such a two-plane, it is generated by two supertwistors  $\mathbf{T}^\mu = (\mathbf{T}_A{}^\mu)$ , with  $\mu = 1, 2$ , such that (i) the bodies<sup>5</sup> of  $\mathbf{T}_{\hat{\alpha}}{}^1$  and  $\mathbf{T}_{\hat{\alpha}}{}^2$  are linearly independent twistors; and (ii) these supertwistors obey the equations

$$\langle \mathbf{T}^\mu, \mathbf{T}^\nu \rangle \equiv \bar{\mathbf{T}}^\mu \mathbf{T}^\nu = 0, \quad \mu, \nu = 1, 2. \quad (3.6)$$

The basis chosen,  $\{\mathbf{T}^\mu\}$ , is defined modulo the equivalence relation

$$\{\mathbf{T}^\mu\} \sim \{\tilde{\mathbf{T}}^\mu\}, \quad \tilde{\mathbf{T}}^\mu = \mathbf{T}^\nu C_\nu{}^\mu, \quad C \in \mathrm{GL}(2, \mathbb{C}). \quad (3.7)$$

The two bases,  $\{\mathbf{T}^\mu\}$  and  $\{\tilde{\mathbf{T}}^\mu\}$ , define one and the same two-plane in  $\mathbb{C}^{4|\mathcal{N}}$ . It may be shown that  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  is a homogeneous space of  $\mathrm{SU}(2, 2|\mathcal{N})$ .

The standard  $\mathcal{N}$ -extended Minkowski superspace,  $\mathbb{M}^{4|4\mathcal{N}}$  or more traditionally  $\mathbb{R}^{4|4\mathcal{N}}$ , can be identified with a certain open domain of  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  on which the upper  $2 \times 2$  matrix block of the supermatrix

$$(\mathbf{T}_A{}^\mu) = \begin{pmatrix} \mathbf{T}_\alpha{}^\mu \\ \mathbf{T}_{\dot{\alpha}}{}^\mu \\ \mathbf{T}_i{}^\mu \end{pmatrix} \quad (3.8)$$

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<sup>4</sup>The case  $\mathcal{N} = 4$  is known to be somewhat special. Since off-shell supersymmetric theories exist for  $\mathcal{N} = 1, 2, 3$ , here we do not dwell on the special features of  $\mathcal{N} = 4$ .

<sup>5</sup>See [24] for the necessary information about infinite dimensional Grassmann algebra  $\Lambda_\infty$ .

is non-singular. The freedom to choose the basis, eq. (3.7), can be used to fix  $T_\alpha{}^\mu = \delta_\alpha{}^\mu$ . In this gauge, the supermatrix  $(\mathbf{T}_A{}^\mu)$  takes the form have

$$(\mathbf{T}_A{}^\mu) = \begin{pmatrix} \delta_\alpha{}^\beta \\ -i x_+^{\dot{\alpha}\beta} \\ 2\theta_i{}^\beta \end{pmatrix}, \quad x_+^{\dot{\alpha}\beta} := x_+^m \tilde{\sigma}_m{}^{\dot{\alpha}\beta}. \quad (3.9)$$

Due to (3.6), the bosonic  $\tilde{x}_+ = (x_+^{\dot{\alpha}\beta})$  and fermionic  $\theta = (\theta_i{}^\beta)$  variables obey the reality condition

$$\tilde{x}_+ - \tilde{x}_- = 4i\theta^\dagger\theta, \quad \tilde{x}_- = (\tilde{x}_+)^{\dagger}. \quad (3.10)$$

It is solved by

$$x_{\pm}^{\dot{\alpha}\beta} = x^{\dot{\alpha}\beta} \pm 2i\bar{\theta}^{\dot{\alpha}i}\theta_i{}^\beta, \quad \bar{\theta}^{\dot{\alpha}i} = (\theta_i{}^\alpha)^*, \quad \tilde{x}^\dagger = \tilde{x}, \quad (3.11)$$

with  $z = (x^a, \theta_i{}^\alpha, \bar{\theta}_\alpha{}^i)$  the coordinates of  $\mathcal{N}$ -extended Minkowski superspace  $\mathbb{R}^{4|4\mathcal{N}}$ . We see that the supertwistors in the Minkowski chart (3.9) are parametrized by the *chiral* coordinates  $x_+^a$  and  $\theta_i{}^\alpha$ . More details on the supertwistor construction for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  can be found, e.g., in [14].

We can elaborate on implications of the construction presented. Using the two supertwistors  $\mathbf{T}^\mu$ , which describe a point of  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$ , we define the following supermatrix

$$\mathbf{Y}_{AB} := \mathbf{T}_A{}^\mu \mathbf{T}_B{}^\nu \varepsilon_{\mu\nu}, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad \varepsilon_{12} = -1. \quad (3.12)$$

It is graded antisymmetric,

$$\mathbf{Y}_{AB} = -(-1)^{\epsilon_A \epsilon_B} \mathbf{Y}_{BA}, \quad (3.13)$$

and defined modulo the equivalence relation

$$\mathbf{Y}_{AB} \sim c \mathbf{Y}_{AB}, \quad c = \det(C_{\hat{\alpha}}{}^{\hat{\beta}}) \in \mathbb{C} - \{0\}. \quad (3.14)$$

The important property of this supermatrix is

$$\mathbf{Y}_{[AB}\mathbf{Y}_{C]D} = 0 \quad \longleftrightarrow \quad \mathbf{Y}_{[AB}\mathbf{Y}_{CD]} = 0, \quad (3.15)$$

where  $[...]$  denotes the graded antisymmetrization of indices.

It should be emphasized that the supermatrix  $\mathbf{Y}_{AB}$  defined by (3.12) is non-zero, and so is the body of its bosonic block  $\mathbf{Y}_{\hat{\alpha}\hat{\beta}}$ . This follows from an easily verified statement: if  $\mathbf{T}_A$  and  $\mathbf{S}_A$  are two supertwistors such that the bodies of  $\mathbf{T}_{\hat{\alpha}}$  and  $\mathbf{S}_{\hat{\alpha}}$  are non-zero, then it holds that

$$\mathbf{T}_{[A}\mathbf{S}_{B]} = 0 \quad \longleftrightarrow \quad \mathbf{T}_A = \lambda \mathbf{S}_A. \quad (3.16)$$

If the bodies of  $\mathbf{T}_{\hat{\alpha}}$  and  $\mathbf{S}_{\hat{\alpha}}$  vanish, however, it is easy to construct two supertwistors  $\mathbf{T}_A$  and  $\mathbf{S}_A$  such that  $\mathbf{T}_{[A}\mathbf{S}_{B]} = 0$  but  $\mathbf{T}_A \neq \lambda \mathbf{S}_A$  for any  $\lambda$ .

In the Minkowski chart, choosing the gauge (3.9) gives

$$\mathbf{Y}_{AB} = \left( \begin{array}{c|c|c} \varepsilon_{\alpha\beta} & -\mathrm{i} x_{+\alpha} \dot{\beta} & 2\theta_{\alpha j} \\ \hline \mathrm{i} x_{+} \dot{\alpha} \beta & \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} x_{+} \dot{\gamma} x_{+\gamma\dot{\gamma}} & -2\mathrm{i} x_{+} \dot{\alpha} \gamma \theta_{\gamma j} \\ \hline -2\theta_{i\beta} & 2\mathrm{i} \theta_{i\gamma} x_{+} \dot{\beta} \gamma & 4\theta_i \theta_j \end{array} \right) . \quad (3.17)$$

Using the dual supertwistors  $\bar{\mathbf{T}}^{\mu}$  allows us to define another supermatrix

$$\bar{\mathbf{Y}}^{AB} := \varepsilon_{\mu\nu} \bar{\mathbf{Y}}^{\mu A} \bar{\mathbf{Y}}^{\nu B} . \quad (3.18)$$

The supermatrices  $\mathbf{Y} = (\mathbf{Y}_{AB})$  and  $\bar{\mathbf{Y}} = (\bar{\mathbf{Y}}^{AB})$  are related to each other as

$$\bar{\mathbf{Y}} = -\Omega \mathbf{Y}^{\dagger} \Omega . \quad (3.19)$$

where the definition of  $\mathbf{Y}^{\dagger}$  is the same as for ordinary matrices. The null conditions (3.6) give

$$\bar{\mathbf{Y}}^{AC} \mathbf{Y}_{CB} = 0 . \quad (3.20)$$

### 3.2 Bi-supertwistor realization

We give an alternative realization of compactified Minkowski superspace  $\bar{\mathbb{M}}^{4|\mathcal{N}}$ . In the space of all complex graded antisymmetric matrices,  $\mathbf{Y}_{AB} = -(-1)^{\epsilon_A \epsilon_B} \mathbf{Y}_{BA}$ , we consider a surface  $\mathfrak{L}$  consisting of those supermatrices which obey the algebraic constraints

$$\mathbf{Y}_{[AB} \mathbf{Y}_{CD]} = 0 , \quad (3.21a)$$

$$\bar{\mathbf{Y}}^{AC} \mathbf{Y}_{CB} = 0 , \quad (3.21b)$$

and satisfy the following condition: for each supermatrix  $\mathbf{Y} \in \mathfrak{L}$ , the *body* of its bosonic block  $\mathbf{Y}_{\hat{\alpha}\hat{\beta}}$  defined by

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_{\hat{\alpha}\hat{\beta}} & \mathbf{Y}_{\hat{\alpha}j} \\ \mathbf{Y}_{i\hat{\beta}} & \mathbf{Y}_{ij} \end{pmatrix} \quad (3.22)$$

is a non-zero antisymmetric  $4 \times 4$  matrix. Our goal is to show that the superspace  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  can be identified with the space of equivalence classes in  $\mathfrak{L}$  with respect to the equivalence relation

$$\mathbf{Y}_{AB} \sim c \mathbf{Y}_{AB}, \quad c \in \mathbb{C} - \{0\}. \quad (3.23)$$

It is natural to think of  $\mathbf{Y}_{AB}$  as a graded two-form on the dual supertwistor space.

Let us first demonstrate that eq. (3.21a) implies that  $\mathbf{Y}_{AB}$  is decomposable provided the body of  $\mathbf{Y}_{\hat{\alpha}\hat{\beta}}$  is non-zero. This means that  $\mathbf{Y}_{AB}$  can be represented as

$$\mathbf{Y}_{AB} = \mathbf{T}_A \mathbf{S}_B - (-1)^{\epsilon_A \epsilon_B} \mathbf{T}_B \mathbf{S}_A, \quad (3.24)$$

for some supertwistors  $\mathbf{T}_A$  and  $\mathbf{S}_A$ . It follows from eq. (3.21a) that

$$\mathbf{T}_{[A} \mathbf{Y}_{BC]} = 0, \quad (3.25)$$

where  $\mathbf{T}_A := \mathbf{V}^B \mathbf{Y}_{BA}$ , for any dual supertwistor  $\mathbf{V}^A$ . Since the body of  $\mathbf{Y}_{\hat{\alpha}\hat{\beta}}$  is non-zero, it is possible to choose  $\mathbf{V}^A$  such that the body of  $\mathbf{T}_{\hat{\alpha}}$  is non-zero. The properties of  $\mathbf{T}_A$  and  $\mathbf{Y}_{AB}$  are such that we can apply a generalization of Cartan's lemma.<sup>6</sup> This generalization states that the condition (3.25) implies the validity of (3.24), for some supertwistor  $\mathbf{S}_A$  such that the body of  $\mathbf{S}_{\hat{\alpha}}$  is non-zero. It is clear that the bodies of  $\mathbf{T}_{\hat{\alpha}}$  and  $\mathbf{S}_{\hat{\alpha}}$  are linearly independent twistors. Since  $\mathbf{Y}_{AB}$  is defined modulo arbitrary re-scalings, eq. (3.23), one can see that the two supertwistors  $\mathbf{T}^\mu := (\mathbf{T}, \mathbf{S})$  are defined modulo the equivalence relation (3.7).

Starting from the graded antisymmetric supermatrix (3.24), we introduce its conjugate  $\bar{\mathbf{Y}}^{AB}$ , eq. (3.19), and compute the left-hand side of (3.21b). Then eq. (3.21b) becomes equivalent to

$$\begin{aligned} \bar{\mathbf{Y}}^{AC} \mathbf{Y}_{CB} = \bar{\mathbf{T}}^A \left\{ \langle \mathbf{S}, \mathbf{T} \rangle \mathbf{S}_B - \langle \mathbf{S}, \mathbf{S} \rangle \mathbf{T}_B \right\} \\ + \bar{\mathbf{S}}^A \left\{ \langle \mathbf{T}, \mathbf{S} \rangle \mathbf{T}_B - \langle \mathbf{T}, \mathbf{T} \rangle \mathbf{S}_B \right\} = 0. \end{aligned} \quad (3.26)$$

Since the bodies of  $\bar{\mathbf{T}}^{\hat{\alpha}}$  and  $\bar{\mathbf{S}}^{\hat{\alpha}}$  are linearly independent, we conclude that the two expressions in figure brackets must vanish. Since the bodies of  $\mathbf{T}_{\hat{\alpha}}$  and  $\mathbf{S}_{\hat{\alpha}}$  are linearly independent, we end up with the null conditions

$$\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{T}, \mathbf{S} \rangle = \langle \mathbf{S}, \mathbf{T} \rangle = \langle \mathbf{S}, \mathbf{S} \rangle = 0. \quad (3.27)$$

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<sup>6</sup>For completeness, we recall the formulation of Cartan's lemma, see e.g. [21, 25]. Consider a finite dimensional vector space  $V$ . Let  $\omega \in \wedge^p V$  be a  $p$ -vector, and  $\varphi \in V$  be a non-zero one-vector such that  $\varphi \wedge \omega = 0$ . Then there exists a  $(p-1)$ -vector  $\eta$  such that  $\omega = \varphi \wedge \eta$ .

As a result, we have demonstrated that the bi-supertwistor realization introduced is completely equivalent to the standard supertwistor realization of  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  described in the previous subsection.

The constraints (3.21a) and (3.21b) for  $\mathcal{N} = 1$  were identified in the third version of [12]. Ref. [13] closely followed the first version of [12] and did not provide the correct constraints. The constraint (3.21a) and a considerable part of the bi-supertwistor construction for general  $\mathcal{N}$ , including eq. (3.17), were known to Siegel in the mid 1990s [31, 32]. The equivalence between the supertwistor and bi-supertwistor formulations was not proved in [12, 31, 32].

## 4 Compactified 4D $\mathcal{N} = 2$ harmonic/projective superspace

As is well known, all  $\mathcal{N} = 2$  supersymmetric theories in four dimensions are naturally formulated in the superspace  $\mathbb{R}^{4|8} \times \mathbb{C}P^1$  introduced for the first time by Rosly [26]. Depending on the specific superfield methods employed to construct off-shell  $\mathcal{N} = 2$  supersymmetric theories, this superspace is called harmonic [27, 28] or projective [29, 30]. Here we start by describing the conformally compactified version of  $\mathbb{R}^{4|8} \times \mathbb{C}P^1$  following Ref. [14] which built on the earlier publications [33, 34, 35]. After that we introduce a new bi-supertwistor realization for this compactified superspace.

Ordinary supertwistors introduced by Ferber [22],

$$(\mathbf{T}_A) = \begin{pmatrix} \mathbf{T}_{\hat{\alpha}} \\ \mathbf{T}_i \end{pmatrix}, \quad (4.1)$$

are characterized by the following Grassmann parities of their components:

$$\epsilon(\mathbf{T}_A) = \epsilon_A = \begin{cases} 0 & A = \hat{\alpha} \\ 1 & A = i \end{cases}. \quad (4.2)$$

Such supertwistors are often called *even*. One can also consider *odd* supertwistors,

$$(\mathbf{\Xi}_A) = \begin{pmatrix} \mathbf{\Xi}_{\hat{\alpha}} \\ \mathbf{\Xi}_i \end{pmatrix}, \quad (4.3)$$

with opposite Grassmann parities

$$\epsilon(\mathbf{\Xi}_A) = 1 + \epsilon_A \pmod{2}. \quad (4.4)$$

Both even and odd supertwistors should be used [33, 34] in order to define harmonic-like superspaces in extended conformal supersymmetry.

Now, we accompany the two even supertwistors  $\mathbf{T}^\mu$ , which occur in the construction of the compactified  $\mathcal{N} = 2$  superspace  $\overline{\mathbb{M}}^{4|8}$ , by an odd supertwistor  $\Xi$  such that the body of  $\Xi_i$  is non-zero. These supertwistors are required to obey the null conditions

$$\langle \mathbf{T}^\mu, \mathbf{T}^\nu \rangle = \langle \mathbf{T}^\mu, \Xi \rangle = 0, \quad \mu, \nu = 1, 2, \quad (4.5)$$

and are defined modulo the equivalence relation

$$(\Xi, \mathbf{T}^\mu) \sim (\Xi, \mathbf{T}^\nu) \begin{pmatrix} d & 0 \\ \rho_\nu & C_\nu^\mu \end{pmatrix}, \quad \begin{pmatrix} d & 0 \\ \rho & C \end{pmatrix} \in \text{GL}(1|2), \quad (4.6)$$

with  $\rho_\nu$  anticommuting complex parameters. The superspace obtained can be seen to be  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$ . It is a homogeneous space of  $\text{SU}(2, 2|2)$ .

To understand the global structure of the superspace introduced, it is convenient to restrict our consideration to its Minkowski chart defined the same way as in the previous section. The freedom to perform equivalence transformations (4.6) allows us to choose  $\mathbf{T}^\mu$  and  $\Xi$  to look like

$$(\mathbf{T}_A{}^\mu) = \begin{pmatrix} \delta_\alpha^\beta \\ -i x_+^{\dot{\alpha}\beta} \\ 2\theta_i^\beta \end{pmatrix}, \quad (\Xi_A) = \begin{pmatrix} 0 \\ 2\bar{\theta}^{\dot{\alpha}j} v_j \\ v_i \end{pmatrix}, \quad v_i \in \mathbb{C}^2 - \{0\}. \quad (4.7)$$

The isotwistor  $v_i$  is defined modulo the equivalence relation

$$v_i \sim d v_i, \quad d \in \mathbb{C} - \{0\}. \quad (4.8)$$

This shows that the superspace under consideration is indeed  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$ .

We are in a position to formulate a bi-supertwistor realization for  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$ . It is given in terms of complex variables  $(\mathbf{Y}_{AB}, \Xi_A)$ , where  $\mathbf{Y}_{AB}$  obeys the conditions given in subsection 3.2, while  $\Xi$  is an odd supertwistor such that (i) the body of  $\Xi_i$  is non-zero;<sup>7</sup> and (ii)  $\Xi$  obeys the condition

$$\bar{\mathbf{Y}}^{AB} \Xi_B = 0. \quad (4.9)$$

The variables  $(\mathbf{Y}_{AB}, \Xi_A)$  are defined modulo the equivalence relation

$$(\Xi_A, \mathbf{Y}_{AB}) \sim (\Xi'_A, \mathbf{Y}'_{AB}) = (\Xi_A, \mathbf{Y}_{AC}) \begin{pmatrix} d & 0 \\ \rho^C & c \delta^C_B \end{pmatrix}, \quad (4.10)$$

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<sup>7</sup>This condition implies that the body of  $\bar{\Xi}^C \Xi_C$  is non-zero, that is  $(\bar{\Xi}^C \Xi_C)^{-1}$  is well defined.

where  $c, d \in \mathbb{C} - \{0\}$ , and  $\rho^C$  is an odd dual supertwistor.

Let us introduce a superfield of the general form ( $n \geq 0$ ):

$$\Phi^{(n)}(\mathbf{Y}, \bar{\mathbf{Y}}, \Xi, \bar{\Xi}) = \sum_{k=0}^{\infty} \frac{\bar{\Xi}^{B_k} \dots \bar{\Xi}^{B_1}}{(\bar{\Xi}^C \Xi_C)^k} \Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k}}(\mathbf{Y}, \bar{\mathbf{Y}}) \Xi_{A_{n+k}} \dots \Xi_{A_1} , \quad (4.11)$$

where the Fourier coefficients  $\Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k}}(\mathbf{Y}, \bar{\mathbf{Y}})$  have the following properties:

(i) they are homogeneous functions of  $\mathbf{Y}$ 's and their conjugates,

$$\Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k}}(c\mathbf{Y}, \bar{c}\bar{\mathbf{Y}}) = c^{-\Delta} \bar{c}^{-\bar{\Delta}} \Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k}}(\mathbf{Y}, \bar{\mathbf{Y}}) , \quad (4.12)$$

for some parameter  $\Delta$  and  $\bar{\Delta}$ ;

(ii) they are graded antisymmetric in their  $A$ -indices and separately in their  $B$ -indices,

$$\Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k}}(\mathbf{Y}, \bar{\mathbf{Y}}) = \Phi_{[B_1 \dots B_k]}^{[A_1 \dots A_{n+k}]}(\mathbf{Y}, \bar{\mathbf{Y}}) ; \quad (4.13)$$

(iii) they are tensors at the point  $(\mathbf{Y}, \bar{\mathbf{Y}})$  of  $\overline{\mathbb{M}}^{4|8}$ , that is

$$\bar{\mathbf{Y}}^{CD} \Phi_{DB_2 \dots B_k}^{A_1 \dots A_{n+k}}(\mathbf{Y}, \bar{\mathbf{Y}}) = 0 , \quad (4.14a)$$

$$\Phi_{B_1 \dots B_k}^{A_1 \dots A_{n+k-1} D}(\mathbf{Y}, \bar{\mathbf{Y}}) \mathbf{Y}_{DC} = 0 . \quad (4.14b)$$

These tensor conditions guarantee that  $\Phi^{(n)}(\mathbf{Y}, \bar{\mathbf{Y}}, \Xi, \bar{\Xi})$  changes under the equivalence transformation (4.10) as follows:

$$\Phi^{(n)}(\mathbf{Y}', \bar{\mathbf{Y}}', \Xi', \bar{\Xi}') = d^n c^{-\Delta} \bar{c}^{-\bar{\Delta}} \Phi^{(n)}(\mathbf{Y}, \bar{\mathbf{Y}}, \Xi, \bar{\Xi}) . \quad (4.15)$$

Eq. (4.11) defines a superconformal Fourier expansion. Without loss of generality, the Fourier coefficients in (4.11) can be subject to an irreducibility condition that a super-trace of any  $A$ -index with a  $B$ -index vanish.

It should be mentioned that there exist two more equivalent realizations for the superspace  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$ , which will be referred to as Type A and Type B formulations. Both realizations are given in terms of complex variables  $(\mathbf{Y}_{AB}, \Xi_A, \Sigma_A)$ , where  $\mathbf{Y}_{AB}$  obeys the conditions given in subsection 3.2, while  $\Xi$  and  $\Sigma$  are odd supertwistors such that (i) the bodies of  $\Xi_i$  and  $\Sigma_i$  are non-zero; and (ii)  $\Xi$  and  $\Sigma$  obey the null conditions

$$\bar{\mathbf{Y}}^{AB} \Xi_B = 0 , \quad \bar{\mathbf{Y}}^{AB} \Sigma_B = 0 ; \quad (4.16)$$

(iii) the odd supertwistor  $\Xi$  is defined modulo arbitrary equivalence transformations (4.10). The two realizations differ in additional conditions (iv) and (v) imposed on  $\Sigma$ . Let us describe these conditions.

In Type A formulation, the odd supertwistor  $\Sigma$  is required to (iv) obey the additional null condition

$$\bar{\Sigma}^B \Xi_B = 0 ; \quad (4.17)$$

as well as (v) be defined modulo the equivalence relation

$$\Sigma_A \sim \Sigma'_A = f \Sigma_A + Y_{AC} \kappa^C , \quad (4.18)$$

where  $f \in \mathbb{C} - \{0\}$ , and  $\kappa^C$  is an arbitrary odd dual supertwistor. It follows that the bodies of  $\Xi_i$  and  $\Sigma_i$  are linearly independent. One can see that no additional degrees of freedom are associated with  $\Sigma$ .

In Type B formulation, the odd supertwistor  $\Sigma$  is such that (iv) the bodies of  $\Xi_i$  and  $\Sigma_i$  are linearly independent; and (v)  $\Sigma$  is defined modulo the equivalence relation<sup>8</sup>

$$\Sigma_A \sim \Sigma'_A = f \Sigma_A + g \Xi_A + Y_{AC} \kappa^C , \quad (4.19)$$

where  $f \in \mathbb{C} - \{0\}$ ,  $g \in \mathbb{C}$ , and  $\kappa^C$  is an arbitrary odd dual supertwistor. As in the previous case of Type A formulation, no degrees of freedom are associated with  $\Sigma$ . Type B formulation is the bi-supertwistor version of the so-called harmonic realization for  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$  introduced in [14].

## 5 Compactified 3D Minkowski space

In the remainder of this paper, we present three-dimensional analogues of the four-dimensional results discussed in sections 2 to 4. The (super)twistor realizations for 3D compactified Minkowski space and its supersymmetric extensions were developed in [15]. Here we will build on the constructions presented in [15]. The interested reader is referred to that paper for more details, including the spinor conventions in  $3 + 2$  dimensions.

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<sup>8</sup>If the equivalence relation (4.19) is replaced by (4.18), then we end up with a space  $\overline{\mathbb{M}}^{4|8} \times T^* \mathbb{CP}^1$ , where  $T^* \mathbb{CP}^1$  denotes the cotangent bundle of  $\mathbb{CP}^1$ . One can think of  $T^* \mathbb{CP}^1$  as the complexification of  $\mathbb{CP}^1$ . The importance of superspace  $\mathbb{R}^{4|8} \times T^* \mathbb{CP}^1$  in the context  $\mathcal{N} = 2$  supersymmetric sigma models has recently been emphasized by Butter [36].



## 5.1 Twistor realization

Consider a symplectic four-dimensional real vector space. We can think of it as  $\mathbb{R}^4$  equipped with a skew-symmetric inner product:

$$\langle T|S\rangle_J := T^T J S \equiv T_{\hat{\alpha}} J^{\hat{\alpha}\hat{\beta}} S_{\hat{\beta}} = -\langle S|T\rangle_J , \quad J = (J^{\hat{\alpha}\hat{\beta}}) = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} , \quad (5.1)$$

for any vectors  $T, S \in \mathbb{R}^4$ . By construction, this inner product is invariant under the group<sup>9</sup>  $\mathbf{Sp}(4, \mathbb{R})$ . We refer to this vector space as the 3D twistor space. Its elements  $T, S \in \mathbb{R}^4$  are called 3D twistors.

The elements of the group  $\mathbf{Sp}(4, \mathbb{R})$  can be represented by  $4 \times 4$  block matrices

$$g = (g_{\hat{\alpha}}^{\hat{\beta}}) = \begin{pmatrix} \mathcal{A} & -\mathcal{B} \\ -\mathcal{C} & \mathcal{D} \end{pmatrix} \in \mathbf{SL}(4, \mathbb{R}) , \quad g^T J g = J , \quad (5.2)$$

where  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  are  $2 \times 2$  matrices. The symplectic group  $\mathbf{Sp}(4, \mathbb{R})$  is the two to one covering of the connected component,  $\mathbf{SO}_0(3, 2)$ , of the conformal group in three dimensions. A twistor looks like

$$T = (T_{\hat{\alpha}}) = \begin{pmatrix} f_{\alpha} \\ h^{\alpha} \end{pmatrix} , \quad (5.3)$$

with the two-component vectors  $f_{\alpha}$  and  $h^{\alpha}$  being real commuting 3D spinors.

A *Lagrangian subspace* of the twistor space is defined to be a maximal isotropic vector subspace of  $\mathbb{R}^4$ . Such a subspace is necessarily two-dimensional. We denote by  $\overline{\mathbb{M}}_{\mathbf{T}}^3$  the space of all Lagrangian subspaces of  $\mathbb{R}^4$ . One can show that  $\overline{\mathbb{M}}_{\mathbf{T}}^3$  is a homogeneous space of the group  $\mathbf{Sp}(4, \mathbb{R})$  and has the structure

$$\overline{\mathbb{M}}_{\mathbf{T}}^3 = \mathbf{U}(2)/\mathbf{O}(2) , \quad (5.4)$$

see, e.g., [15, 37] for technical details.

Conformally compactified 3D Minkowski space can be identified with  $\overline{\mathbb{M}}_{\mathbf{T}}^3$ . Indeed, given a Lagrangian subspace, it is generated by two linearly independent twistors  $T^{\mu}$ , with  $\mu = 1, 2$ , such that

$$\langle T^1|T^2\rangle_J = 0 . \quad (5.5)$$

Obviously, the basis chosen,  $\{T^{\mu}\}$ , is defined only modulo the equivalence relation

$$\{T^{\mu}\} \sim \{\tilde{T}^{\mu}\} , \quad \tilde{T}^{\mu} = T^{\nu} R_{\nu}^{\mu} , \quad R \in \mathbf{GL}(2, \mathbb{R}) . \quad (5.6)$$

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<sup>9</sup>This group was denoted  $\mathbf{Sp}(2, \mathbb{R})$  in [15].

Minkowski space  $\mathbb{M}^3 \equiv \mathbb{R}^{2,1}$  can be identified with an open dense subset of  $\overline{\mathbb{M}}^3_{\mathbb{T}}$  consisting of those Lagrangian subspaces which are described by  $4 \times 2$  matrices of the form:

$$(T_{\hat{\alpha}}{}^{\mu}) = \begin{pmatrix} F_{\alpha}{}^{\mu} \\ H^{\alpha\mu} \end{pmatrix}, \quad \det F \neq 0. \quad (5.7)$$

In accordance with the equivalence relation (5.6), we can choose  $F = \mathbb{1}_2$ . Then the null condition gives

$$(T_{\hat{\alpha}}{}^{\mu}) = \begin{pmatrix} \mathbb{1}_2 \\ -x \end{pmatrix}, \quad x^{\mathbb{T}} = x = (x^{\alpha\beta}) \in \text{Mat}(2, \mathbb{R}). \quad (5.8)$$

This is a standard matrix realization of 3D Minkowski space. The conformal transformation (5.2) acts on  $x$  as follows:

$$x' = (C + Dx)(A + Bx)^{-1}. \quad (5.9)$$

The Poincaré group corresponds to the subgroup of  $\text{Sp}(4, \mathbb{R})$  consisting of the matrices

$$g = \left( \begin{array}{c|c} M & 0 \\ \hline -aM & (M^{-1})^{\mathbb{T}} \end{array} \right), \quad a = a^{\mathbb{T}} \in \text{Mat}(2, \mathbb{R}), \quad M \in \text{SL}(2, \mathbb{R}). \quad (5.10)$$

## 5.2 Bi-twistor realization

We now describe an alternative, *bi-twistor* realization of  $\overline{\mathbb{M}}^3$ . Let us denote by  $\mathcal{L}$  the set of all *non-zero* real antisymmetric matrices  $X_{\hat{\alpha}\hat{\beta}} = -X_{\hat{\beta}\hat{\alpha}}$  which obey the algebraic constraints

$$X_{[\hat{\alpha}\hat{\beta}}X_{\hat{\gamma}\hat{\delta}]} = 0, \quad (5.11a)$$

$$X_{\hat{\alpha}\hat{\gamma}}J^{\hat{\gamma}\hat{\delta}}X_{\hat{\delta}\hat{\beta}} = 0, \quad (5.11b)$$

$$J^{\hat{\beta}\hat{\alpha}}X_{\hat{\alpha}\hat{\beta}} = 0. \quad (5.11c)$$

We introduce the quotient space  $\overline{\mathbb{M}}^3_{\text{BT}} = \mathcal{L}/\sim$ , where the equivalence relation is defined by

$$X_{\hat{\alpha}\hat{\beta}} \sim r X_{\hat{\alpha}\hat{\beta}}, \quad r \in \mathbb{R} - \{0\}. \quad (5.12)$$

The twistor and bi-twistor realizations are related to each other as follows:

$$X_{\hat{\alpha}\hat{\beta}} := T_{\hat{\alpha}}{}^{\mu}T_{\hat{\beta}}{}^{\nu}\varepsilon_{\mu\nu}, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad \varepsilon_{12} = -1. \quad (5.13)$$

The proof is left to the reader as an exercise.

## 6 Compactified 3D Minkowski superspace

Consider the graded symplectic metric on  $\mathbb{R}^{4|\mathcal{N}}$

$$\mathbf{J} = (\mathbf{J}^{AB}) = \begin{pmatrix} J^{\hat{\alpha}\hat{\beta}} & 0 \\ 0 & \mathbf{i}\delta^{IJ} \end{pmatrix}, \quad I, J = 1, \dots, \mathcal{N}. \quad (6.1)$$

The 3D  $\mathcal{N}$ -extended superconformal group  $\text{OSp}(\mathcal{N}|4, \mathbb{R})$  consists of supermatrices  $g$  of the form

$$g^{\text{sT}} \mathbf{J} g = \mathbf{J}, \quad g = \begin{pmatrix} A \parallel B \\ \hline C \parallel D \end{pmatrix}, \quad g^{\text{sT}} = \begin{pmatrix} A^{\text{T}} \parallel -C^{\text{T}} \\ \hline B^{\text{T}} \parallel D^{\text{T}} \end{pmatrix}. \quad (6.2)$$

Here the even matrices  $A, D$  and the odd matrix  $B$  have real matrix elements, while the odd matrix  $C$  has purely imaginary matrix elements. Supermatrices  $g$  of this type are called *real*, in accordance with [24].

The superconformal group  $\text{OSp}(\mathcal{N}|4, \mathbb{R})$  naturally acts on supertwistor space  $\mathbb{R}^{4|\mathcal{N}}$  spanned by elements of the form

$$\mathbf{T} \equiv (\mathbf{T}_A) = \begin{pmatrix} \mathbf{T}_{\hat{\alpha}} \\ \mathbf{i}\varphi_I \end{pmatrix} = \begin{pmatrix} f_{\alpha} \\ h^{\alpha} \\ \mathbf{i}\varphi_I \end{pmatrix}, \quad \epsilon(\mathbf{T}_A) = \epsilon_A = \begin{cases} 0 & A = \hat{\alpha} \\ 1 & A = I \end{cases} \quad (6.3)$$

and endowed with the graded symplectic two-form  $\mathcal{J} = \frac{1}{2} \mathbf{J}^{AB} d\mathbf{T}_B \wedge d\mathbf{T}_A$ . This action preserves  $\mathcal{J}$ , and thus the symplectic inner product on  $\mathbb{R}^{4|\mathcal{N}}$  defined by

$$\langle \mathbf{T} | \mathbf{S} \rangle_{\mathbf{J}} := \mathbf{T}^{\text{sT}} \mathbf{J} \mathbf{S} = -\langle \mathbf{S} | \mathbf{T} \rangle_{\mathbf{J}}, \quad \mathbf{T}^{\text{sT}} = (\mathbf{T}_{\hat{\alpha}}, -\mathbf{i}\varphi_I), \quad (6.4)$$

with the graded symplectic matrix  $\mathbf{J}$  defined in (6.1). Any element  $\mathbf{T} \in \mathbb{R}^{4|\mathcal{N}}$  is called an *even real supertwistor*.

### 6.1 Supertwistor realization

A *Lagrangian subspace* of  $\mathbb{R}^{4|\mathcal{N}}$  is defined to be a maximal isotropic subspace of the supertwistor space [15]. We denote by  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  the space of all Lagrangian subspaces of  $\mathbb{R}^{4|\mathcal{N}}$ . Given such a subspace, it is generated by two supertwistors  $\mathbf{T}^{\mu}$  such that

- (i) the bodies of  $\mathbf{T}_{\hat{\alpha}}^1$  and  $\mathbf{T}_{\hat{\alpha}}^2$  are linearly independent twistors;
- (ii)  $\mathbf{T}^1$  and  $\mathbf{T}^2$  obey the null condition

$$\langle \mathbf{T}^1 | \mathbf{T}^2 \rangle_{\mathbf{J}} = 0; \quad (6.5)$$

(iii)  $\mathbf{T}^1$  and  $\mathbf{T}^2$  are defined only modulo the equivalence relation

$$\{\mathbf{T}^\mu\} \sim \{\tilde{\mathbf{T}}^\mu\}, \quad \tilde{\mathbf{T}}^\mu = \mathbf{T}^\nu R_\nu{}^\mu, \quad R \in \mathrm{GL}(2, \mathbb{R}). \quad (6.6)$$

A dense open subset  $\mathbb{M}^{3|2\mathcal{N}}$  of  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  consists of those Lagrangian subspaces which are described by supermatrices of the form

$$(\mathbf{T}_A{}^\mu) = \begin{pmatrix} F_\alpha{}^\mu \\ H^{\alpha\mu} \\ i\Upsilon_I{}^\mu \end{pmatrix}, \quad \det(F_\alpha{}^\mu) \neq 0. \quad (6.7)$$

Using the equivalence relation (6.6) allows us to choose  $F = \mathbb{1}_2$ , and hence

$$(\mathbf{T}_A{}^\mu) = \begin{pmatrix} \delta_\alpha{}^\beta \\ -x^{\alpha\beta} + \frac{i}{2}\varepsilon^{\alpha\beta}\theta^2 \\ i\sqrt{2}\theta_I{}^\beta \end{pmatrix}, \quad x^{\alpha\beta} = x^{\beta\alpha}, \quad \theta^2 := \theta_I^\alpha \theta_{\alpha I}. \quad (6.8)$$

Here the bosonic  $x^{\alpha\beta}$  and fermionic  $\theta_I^\alpha \equiv \theta_I{}^\alpha$  parameters are real. Therefore, the subset  $\mathbb{M}^{3|2\mathcal{N}} \subset \overline{\mathbb{M}}^{3|2\mathcal{N}}$  introduced can be identified with  $\mathbb{R}^{3|2\mathcal{N}}$ . This subset is a transformation space of the 3D  $\mathcal{N}$ -extended super-Poincaré group,  $\mathfrak{P}(3|\mathcal{N})$ , which is a subgroup of the superconformal group  $\mathrm{OSp}(\mathcal{N}|2, \mathbb{R})$ , eq. (6.2), spanned by group elements of the form:

$$g = s(a, \epsilon) h(M), \quad (6.9a)$$

$$s(a, \epsilon) = \left( \begin{array}{c|c|c} \delta_\alpha{}^\beta & 0 & 0 \\ \hline -a^{\alpha\beta} + \frac{i}{2}\varepsilon^{\alpha\beta}\epsilon^2 & \delta^\alpha{}_\beta & -\sqrt{2}\epsilon^\alpha{}_J \\ \hline i\sqrt{2}\epsilon_I{}^\beta & 0 & \delta_{IJ} \end{array} \right), \quad (6.9b)$$

$$h(M) = \left( \begin{array}{c|c|c} M & 0 & 0 \\ \hline 0 & (M^{-1})^T & 0 \\ \hline 0 & 0 & \mathbb{1}_\mathcal{N} \end{array} \right), \quad M \in \mathrm{SL}(2, \mathbb{R}). \quad (6.9c)$$

In eq. (6.9b), the bosonic ( $a^{\alpha\beta} = a^{\beta\alpha}$ ) and fermionic ( $\epsilon_I^\alpha = \epsilon^\alpha{}_I \equiv \epsilon_I^\alpha$ ) parameters are real. Evaluating the action of  $\mathfrak{P}(3|\mathcal{N})$  on  $\mathbb{M}^{3|2\mathcal{N}}$  shows that this space is 3D  $\mathcal{N}$ -extended Minkowski superspace.

## 6.2 Bi-supertwistor realization

We give an alternative, *bi-supertwistor* realization of compactified Minkowski superspace  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ . In the space of all *real* graded antisymmetric matrices,  $\mathbf{X}_{AB} =$

$-(-1)^{\epsilon_A \epsilon_B} \mathbf{X}_{BA}$ , we consider a surface  $\mathfrak{L}$  consisting of those supermatrices which obey the algebraic constraints

$$\mathbf{X}_{[AB} \mathbf{X}_{CD]} = 0 , \quad (6.10a)$$

$$(-1)^{\epsilon_C} \mathbf{X}_{AC} \mathbf{J}^{CD} \mathbf{X}_{DB} = 0 , \quad (6.10b)$$

$$\mathbf{J}^{BA} \mathbf{X}_{AB} = 0, \quad (6.10c)$$

and satisfy the additional condition: for each supermatrix  $\mathbf{X} \in \mathfrak{L}$ , the *body* of its bosonic block  $\mathbf{X}_{\hat{\alpha}\hat{\beta}}$  defined by

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{\hat{\alpha}\hat{\beta}} & \mathbf{X}_{\hat{\alpha}J} \\ \mathbf{X}_{I\hat{\beta}} & \mathbf{X}_{IJ} \end{pmatrix} \quad (6.11)$$

is a non-zero antisymmetric  $4 \times 4$  matrix. The supermatrix  $\mathbf{X}$  must be real in the following sense [24]

$$(\mathbf{X}_{AB})^* = (-1)^{\epsilon_A + \epsilon_B + \epsilon_A \epsilon_B} \mathbf{X}_{AB} . \quad (6.12)$$

It turns out that compactified Minkowski superspace can be identified with the quotient space  $\overline{\mathbb{M}}_{\text{BT}}^{3|2\mathcal{N}} = \mathfrak{L} / \sim$ , where the equivalence relation is defined by

$$\mathbf{X}_{AB} \sim r \mathbf{X}_{AB} , \quad r \in \mathbb{R} - \{0\} . \quad (6.13)$$

The supertwistor and bi-supertwistor realizations are related to each other as follows:

$$\mathbf{X}_{AB} := \mathbf{T}_A{}^\mu \mathbf{T}_B{}^\nu \varepsilon_{\mu\nu} , \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} , \quad \varepsilon_{12} = -1 . \quad (6.14)$$

This can be proved in complete analogy with the four-dimensional case considered in section 3.

In the Minkowski chart, choosing the gauge (6.8) gives

$$\mathbf{X}_{AB} = \left( \begin{array}{c|c|c} \varepsilon_{\alpha\beta} & -x_\alpha{}^\beta - \frac{i}{2} \delta_\alpha{}^\beta \theta^2 & i\sqrt{2} \theta_{\alpha J} \\ \hline x^\alpha{}_\beta + \frac{i}{2} \delta^\alpha{}_\beta \theta^2 & \frac{1}{2} \varepsilon^{\alpha\beta} \left( x^{\gamma\delta} x_{\gamma\delta} + \frac{1}{2} (\theta^2)^2 \right) & -i\sqrt{2} x^{\alpha\gamma} \theta_{\gamma J} - \frac{1}{\sqrt{2}} \theta^\alpha{}_J \theta^2 \\ \hline -i\sqrt{2} \theta_{I\beta} & i\sqrt{2} \theta_{I\gamma} x^{\gamma\beta} + \frac{1}{\sqrt{2}} \theta_I{}^\beta \theta^2 & -2\theta_I \theta_J \end{array} \right) . \quad (6.15)$$

## 7 Compactified $\mathcal{N}$ -extended harmonic/projective superspaces in three dimensions

In [16, 15], new homogeneous spaces of the superconformal group  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$  were constructed that include  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  as a submanifold. Their general structure is  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$ , where  $\mathbb{X}_m^{\mathcal{N}}$  is a compact manifold, for any integer  $0 < m \leq [\mathcal{N}/2]^{10}$ . Such superspaces are nontrivial and have interesting applications for  $\mathcal{N} > 2$ . The supertwistor formulation for  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  given in [15] makes use of both even and odd supertwistors. An *odd* supertwistor looks like

$$\Sigma = (\Sigma_A) = \begin{pmatrix} \Sigma_{\hat{\alpha}} \\ \Sigma_I \end{pmatrix}, \quad \epsilon(\Sigma_A) = 1 + \epsilon_A \pmod{2}. \quad (7.1)$$

This supertwistor is called real if all the components  $\Sigma_A$  are real. The super-transpose of  $\Sigma$  is defined to coincide with the ordinary transpose,

$$\Sigma^{\text{sT}} = (\Sigma_{\hat{\alpha}}, \Sigma_I), \quad (7.2)$$

compare with eqs. (6.3) and (6.4). Here we present a bi-supertwistor formulation for the superspaces  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$ , but first we recall their supertwistor realization [15].

Along with the two linearly independent even real supertwistors  $\mathbf{T}^1$  and  $\mathbf{T}^2$  obeying the null condition (6.5), we also consider  $m$  odd *complex* supertwistors  $\Sigma^{\underline{i}}$ , with  $\underline{i} = 1, \dots, m$ , such that (i) the bodies of  $\Sigma^{\underline{i}}$  are linearly independent; (ii) any linear combination of the supertwistors  $\mathbf{T}^{\mu}$  and  $\Sigma^{\underline{i}}$  is null, that is

$$\langle \mathbf{T}^{\mu} | \mathbf{T}^{\nu} \rangle_{\mathbf{J}} = \langle \mathbf{T}^{\mu} | \Sigma^{\underline{j}} \rangle_{\mathbf{J}} = \langle \Sigma^{\underline{i}} | \Sigma^{\underline{j}} \rangle_{\mathbf{J}} = 0. \quad (7.3)$$

The supertwistors  $\mathbf{T}^{\mu}$  and  $\Sigma^{\underline{i}}$  are defined modulo the equivalence relation

$$(\mathbf{T}^{\mu}, \Sigma^{\underline{i}}) \sim (\mathbf{T}^{\nu}, \Sigma^{\underline{j}}) \left( \frac{R_{\nu}^{\mu} \parallel B_{\nu}^{\underline{i}}}{0 \parallel D_{\underline{j}}^{\underline{i}}} \right), \quad \left( \frac{R \parallel B}{0 \parallel D} \right) \in \text{GL}(2|m, \mathbb{C}), \quad R \in \text{GL}(2, \mathbb{R}). \quad (7.4)$$

We emphasize that the fermionic  $B_{\nu}^{\underline{i}}$  and bosonic  $D_{\underline{j}}^{\underline{i}}$  matrix elements are complex. The space  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  is defined to consist of the equivalence classes associated with all possible  $(\mathbf{T}^{\mu}, \Sigma^{\underline{i}})$  under the above conditions.

It is necessary to point out two important features of the construction under consideration. Firstly, the invariant inner product  $\langle \ , \ \rangle_{\mathbf{J}}$  possesses the property

$$\langle \mathcal{T}_1 | \mathcal{T}_2 \rangle_{\mathbf{J}} = -(-1)^{\epsilon_1 \epsilon_2} \langle \mathcal{T}_2 | \mathcal{T}_1 \rangle_{\mathbf{J}}, \quad (7.5)$$

---

<sup>10</sup>As usual, the notation  $[\mathcal{N}/2]$  is used for the integer part of  $\mathcal{N}/2$ .

where  $\epsilon_{1,2}$  denotes the Grassmann parity of  $\mathcal{T}_{1,2}$ . Secondly, associated with the odd supertwistors  $\Sigma^{\dot{i}} = (\Sigma_A^{\dot{i}})$  are their complex conjugates  $\bar{\Sigma}^{\dot{i}} = (\bar{\Sigma}_A^{\dot{i}})$  which possess analogous properties

$$\langle T^\mu | \bar{\Sigma}^{\dot{j}} \rangle_{\mathbf{J}} = \langle \bar{\Sigma}^{\dot{i}} | \bar{\Sigma}^{\dot{j}} \rangle_{\mathbf{J}} = 0 . \quad (7.6)$$

It can be seen that the  $2m$  supertwistors  $\Sigma^{\dot{i}}$  and  $\bar{\Sigma}^{\dot{j}}$  are linearly independent,

$$\det \langle \Sigma^{\dot{i}} | \bar{\Sigma}^{\dot{j}} \rangle_{\mathbf{J}} \neq 0 . \quad (7.7)$$

We are prepared to introduce a bi-supertwistor realization for  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$ . It is given in terms of pairs  $(\mathbf{X}_{AB}, \Sigma_A^{\dot{i}})$ , where the bi-supertwistor  $\mathbf{X}_{AB}$  obeys the conditions formulated in subsection 6.2. As to the odd supertwistors  $\Sigma^{\dot{i}}$ , they must be such that (i) the body of the  $\mathcal{N} \times m$  matrix  $\Sigma_I^{\dot{j}}$  has rank  $m$ ; and (ii) the null conditions hold

$$(-1)^{\epsilon_B} \mathbf{X}_{AB} \mathbf{J}^{BC} \Sigma_C^{\dot{i}} = \Sigma_B^{\dot{i}} \mathbf{J}^{BC} \mathbf{X}_{CA} = 0 , \quad \Sigma_A^{\dot{i}} \mathbf{J}^{AB} \Sigma_B^{\dot{j}} = 0 . \quad (7.8)$$

The superspace  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  is obtained by factorizing the space of all such pairs  $(\mathbf{X}_{AB}, \Sigma_A^{\dot{i}})$  with respect to the equivalence relation

$$(\mathbf{X}_{AB}, \Sigma_A^{\dot{i}}) \sim (\tilde{\mathbf{X}}_{AB}, \tilde{\Sigma}_A^{\dot{i}}) , \quad (7.9a)$$

where

$$\tilde{\mathbf{X}}_{AB} = r \mathbf{X}_{AB} , \quad r \in \mathbb{R} - \{0\} , \quad (7.9b)$$

$$\tilde{\Sigma}_A^{\dot{i}} = (-1)^{\epsilon_B} \mathbf{X}_{AB} \mathbf{J}^{BC} \Xi_C^{\dot{i}} + \Sigma_A^{\dot{j}} D_{\dot{j}}^{\dot{i}} , \quad D \in \text{GL}(2, \mathbb{C}) , \quad (7.9c)$$

where  $\Xi^{\dot{i}}$  are two arbitrary odd supertwistors. This realization for  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  is clearly equivalent to the supertwistor one.

## 8 Conclusion

The bi-supertwistor realization for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  is a natural supersymmetric extension of Dirac's projective lightcone construction for compactified Minkowski space. It was Ferrara [7] who posed the problem of developing such a supersymmetric extension back in 1974. The problem has finally been solved. The supertwistor realization for  $\overline{\mathbb{M}}^{4|4\mathcal{N}}$  can be viewed as a square root of the bi-supertwistor one.

As shown in [12] (see also [38]) the bi-supertwistor realization for  $\overline{\mathbb{M}}^{4|4}$  allows one to derive compact expressions for certain correlation functions in  $\mathcal{N} = 1$  superconformal field theories. The adequate superspace setting for  $\mathcal{N} = 2$  supersymmetric

theories is known to be not the conventional Minkowski superspace  $\mathbb{R}^{4|8}$ , but rather its harmonic/projective extension  $\mathbb{R}^{4|8} \times \mathbb{CP}^1$ . We believe that the bi-supertwistor realization for  $\overline{\mathbb{M}}^{4|8} \times \mathbb{CP}^1$  proposed in section 4 will be useful for (i) the calculation of correlation functions in  $\mathcal{N} = 2$  superconformal field theories; and (ii) the construction of a manifestly  $\text{SU}(2, 2|2)$  invariant formulation for  $\mathcal{N} = 2$  superconformal field theories (compare, e.g., with non-supersymmetric approaches [39, 40]). The superconformal Fourier expansion (4.11) is expected to be especially important in this context.

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## A Spinors in 4 + 2 dimensions

In this appendix we collect the salient information about spinors in 4 + 2 dimensions. A similar discussion can be found, e.g., in [9].

The gamma matrices in 4 + 2 dimensions,  $\Gamma_{\hat{a}}$ , obey the anti-commutation relations

$$\{\Gamma_{\hat{a}}, \Gamma_{\hat{b}}\} = -2\eta_{\hat{a}\hat{b}}\mathbb{1}_8, \quad \eta_{\hat{a}\hat{b}} = \text{diag}(-1, +1, +1, +1, +1, -1) \quad (\text{A.1})$$

and can be chosen to look like

$$\Gamma_{\hat{a}} = \begin{pmatrix} 0 & \Sigma_{\hat{a}} \\ \tilde{\Sigma}_{\hat{a}} & 0 \end{pmatrix}, \quad (\text{A.2})$$

where the  $4 \times 4$  matrices  $\Sigma_{\hat{a}}$  and  $\tilde{\Sigma}_{\hat{a}}$  have the explicit form

$$\Sigma_{\hat{a}} = (\Sigma_a, \Sigma_5, \Sigma_6) = (i\gamma_a, \gamma_5, \mathbb{1}_4) \equiv (\Sigma_{\hat{a}})_{\hat{\alpha}\hat{\beta}}, \quad (\text{A.3a})$$

$$\tilde{\Sigma}_{\hat{a}} = (\tilde{\Sigma}_a, \tilde{\Sigma}_5, \tilde{\Sigma}_6) = (-i\gamma_a, -\gamma_5, \mathbb{1}_4) \equiv (\tilde{\Sigma}_{\hat{a}})^{\hat{\alpha}\hat{\beta}}. \quad (\text{A.3b})$$

Here  $\gamma_a$  are the gamma matrices in 3 + 1 dimensions, and  $\gamma_5 := -i\gamma_0\gamma_1\gamma_2\gamma_3$ . Our choice of  $\gamma_a$  coincides with that adopted in [41, 24], specifically

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad (\text{A.4})$$

where

$$\sigma_a = (\mathbb{1}_2, \vec{\sigma}) \equiv (\sigma_a)_{\alpha\dot{\alpha}}, \quad \tilde{\sigma}_a = (\mathbb{1}_2, -\vec{\sigma}) \equiv (\tilde{\sigma}_a)^{\dot{\alpha}\alpha}. \quad (\text{A.5})$$



The matrices (A.3) obey the relations

$$\Sigma_{\hat{a}}\tilde{\Sigma}_{\hat{b}} + \Sigma_{\hat{b}}\tilde{\Sigma}_{\hat{a}} = -2\eta_{\hat{a}\hat{b}}\mathbb{1}_4 , \quad \tilde{\Sigma}_{\hat{a}}\Sigma_{\hat{b}} + \tilde{\Sigma}_{\hat{b}}\Sigma_{\hat{a}} = -2\eta_{\hat{a}\hat{b}}\mathbb{1}_4 . \quad (\text{A.6})$$

For the matrix  $\Gamma_7 := -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5\Gamma_6$  we obtain

$$\Gamma_7 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix} . \quad (\text{A.7})$$

The Dirac spinor representation of the double covering group of  $\text{SO}_0(4, 2)$  is generated by

$$\mathfrak{J}_{\hat{a}\hat{b}} := -\frac{1}{4}[\Gamma_{\hat{a}}, \Gamma_{\hat{b}}] = \begin{pmatrix} \Sigma_{\hat{a}\hat{b}} & 0 \\ 0 & \tilde{\Sigma}_{\hat{a}\hat{b}} \end{pmatrix} , \quad (\text{A.8})$$

where

$$\Sigma_{\hat{a}\hat{b}} := -\frac{1}{4}(\Sigma_{\hat{a}}\tilde{\Sigma}_{\hat{b}} - \Sigma_{\hat{b}}\tilde{\Sigma}_{\hat{a}}) \equiv (\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}}^{\hat{\beta}} , \quad (\text{A.9a})$$

$$\tilde{\Sigma}_{\hat{a}\hat{b}} := -\frac{1}{4}(\tilde{\Sigma}_{\hat{a}}\Sigma_{\hat{b}} - \tilde{\Sigma}_{\hat{b}}\Sigma_{\hat{a}}) \equiv (\tilde{\Sigma}_{\hat{a}\hat{b}})^{\hat{\alpha}}_{\hat{\beta}} . \quad (\text{A.9b})$$

The matrices  $\Sigma_{\hat{a}\hat{b}}$  are the generators of the group  $\text{SU}(2, 2)$  defined by (2.9). The following isomorphism holds:  $\text{SO}_0(4, 2) \cong \text{SU}(2, 2)/\mathbb{Z}_2$ .

The Hermitian conjugation properties of the gamma matrices are

$$(\Gamma_{\hat{a}})^\dagger = \Gamma_0\Gamma_6\Gamma_{\hat{a}}\Gamma_0\Gamma_6 , \quad (\text{A.10})$$

hence

$$(\mathfrak{J}_{\hat{a}\hat{b}})^\dagger = \Gamma_0\Gamma_6\mathfrak{J}_{\hat{a}\hat{b}}\Gamma_0\Gamma_6 , \quad (\text{A.11})$$

This implies

$$(\Sigma_{\hat{a}})^\dagger = \gamma_0\tilde{\Sigma}_{\hat{a}}\gamma_0 , \quad (\tilde{\Sigma}_{\hat{a}})^\dagger = \gamma_0\Sigma_{\hat{a}}\gamma_0 , \quad (\text{A.12})$$

and hence

$$(\Sigma_{\hat{a}\hat{b}})^\dagger = -\gamma_0\Sigma_{\hat{a}\hat{b}}\gamma_0 , \quad (\tilde{\Sigma}_{\hat{a}\hat{b}})^\dagger = \gamma_0\tilde{\Sigma}_{\hat{a}\hat{b}}\gamma_0 . \quad (\text{A.13})$$

It can be seen that  $\gamma_0$  coincides with the matrix  $\Omega$  in eqs. (2.7) and (2.9).

Given a Dirac spinor

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} , \quad \psi = (\psi_{\hat{\alpha}}) , \quad \phi = (\phi^{\hat{\alpha}}) . \quad (\text{A.14})$$

its Dirac conjugate is defined as follows:

$$\overline{\Psi} := -i \Psi^\dagger \Gamma_0 \Gamma_6 = (\psi^\dagger \gamma_0, -\phi^\dagger \gamma_0) , \quad \psi^\dagger \gamma_0 \equiv (\bar{\psi}^{\hat{\alpha}}) , \quad \phi^\dagger \gamma_0 \equiv (\bar{\phi}_{\hat{\alpha}}) . \quad (\text{A.15})$$

The infinitesimal  $\text{SO}(4, 2)$  transformation laws of these spinors are:

$$\delta \Psi = \frac{1}{2} \omega^{\hat{a}\hat{b}} \mathfrak{J}_{\hat{a}\hat{b}} \Psi , \quad (\text{A.16a})$$

$$\delta \overline{\Psi} = -\frac{1}{2} \overline{\Psi} \omega^{\hat{a}\hat{b}} \mathfrak{J}_{\hat{a}\hat{b}} . \quad (\text{A.16b})$$

The Dirac spinor representation is a sum of two irreducible ones, one of which is the twistor representation and the second is equivalent to its dual (contragredient). The twistor representation is associated with spinors of the form

$$\Psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix} , \quad \psi = (\psi_{\hat{\alpha}}) \quad (\text{A.17})$$

such that  $\Gamma_7 \Psi_L = \Psi_L$ . The Dirac conjugate of  $\Psi_L$ ,

$$\overline{\Psi}_L = (\bar{\psi}, 0) , \quad \bar{\psi} := \psi^\dagger \gamma_0 = (\bar{\psi}^{\hat{\alpha}}) \quad (\text{A.18})$$

transforms according to the dual twistor representation. The infinitesimal  $\text{SO}(4, 2)$  transformation laws of  $\Psi_L$  and  $\overline{\Psi}_L$  are:

$$\delta \psi_{\hat{\alpha}} = \frac{1}{2} \omega^{\hat{c}\hat{d}} (\Sigma_{\hat{c}\hat{d}})_{\hat{\alpha}}^{\hat{\beta}} \psi_{\hat{\beta}} , \quad (\text{A.19a})$$

$$\delta \bar{\psi}^{\hat{\alpha}} = -\frac{1}{2} \bar{\psi}^{\hat{\beta}} \omega^{\hat{c}\hat{d}} (\Sigma_{\hat{c}\hat{d}})_{\hat{\beta}}^{\hat{\alpha}} . \quad (\text{A.19b})$$

Explicit calculations give

$$\frac{1}{2} \omega^{\hat{a}\hat{b}} \Sigma_{\hat{a}\hat{b}} = \left( \begin{array}{c|c} \frac{1}{2} \omega^{ab} \sigma_{ab} - \tau \mathbb{1}_2 & -i b^a \sigma_a \\ \hline -i a^a \tilde{\sigma}_a & \frac{1}{2} \omega^{ab} \tilde{\sigma}_{ab} + \tau \mathbb{1}_2 \end{array} \right) , \quad (\text{A.20})$$

where the parameters  $\tau = \frac{1}{2} \omega^{56}$ ,  $a^a = \frac{1}{2} (\omega^{a6} - \omega^{a5})$  and  $b^a = \frac{1}{2} (\omega^{a6} + \omega^{a5})$  generate a dilatation, a space-time translation and a special conformal transformation respectively. As usual, the  $2 \times 2$  matrices  $\sigma_{ab}$  and  $\tilde{\sigma}_{ab}$  denote the Lorentz generators of the  $(1/2, 0)$  and  $(0, 1/2)$  representations of the Lorentz group in four dimensions [24],

$$\sigma_{ab} := -\frac{1}{4} (\sigma_a \tilde{\sigma}_b - \sigma_b \tilde{\sigma}_a) , \quad \tilde{\sigma}_{ab} := -\frac{1}{4} (\tilde{\sigma}_a \sigma_b - \tilde{\sigma}_b \sigma_a) . \quad (\text{A.21})$$

Since the matrices  $\Gamma_{\hat{a}}$  and  $-\Gamma_{\hat{a}}^T$  constitute two representations of the same Clifford algebra, eq. (A.1), and these representations are necessarily equivalent, we have

$$\mathfrak{C}^{-1} \Gamma_{\hat{a}} \mathfrak{C} = -\Gamma_{\hat{a}}^T , \quad (\text{A.22})$$

hence

$$\mathfrak{C}^{-1} \mathfrak{J}_{\hat{a}\hat{b}} \mathfrak{C} = -\mathfrak{J}_{\hat{a}\hat{b}}^{\text{T}} , \quad (\text{A.23})$$

for some charge conjugation matrix  $\mathfrak{C}$ . It can be chosen as

$$\mathfrak{C} = \begin{pmatrix} 0 & \gamma_5 C \\ -\gamma_5 C & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \mathfrak{C}_{\hat{\alpha}}^{\hat{\beta}} \\ \mathfrak{C}^{\hat{\alpha}}_{\hat{\beta}} & 0 \end{pmatrix} , \quad (\text{A.24})$$

with  $C$  the charge conjugation matrix in  $3+1$  dimensions, which is defined by

$$C^{-1} \gamma_a C = -\gamma_a^{\text{T}} \quad \longrightarrow \quad C^{-1} \gamma_5 C = \gamma_5^{\text{T}} \quad (\text{A.25})$$

and can be chosen as

$$C = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} , \quad C^{-1} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} . \quad (\text{A.26})$$

Using the properties  $C^\dagger = C^{\text{T}} = -C = C^{-1}$  gives

$$\mathfrak{C}^{\text{T}} = \mathfrak{C} . \quad (\text{A.27})$$

The inverse of  $\mathfrak{C}$  is

$$\mathfrak{C}^{-1} = \begin{pmatrix} 0 & (\mathfrak{C}^{-1})^{\hat{\alpha}}_{\hat{\beta}} \\ (\mathfrak{C}^{-1})_{\hat{\alpha}}^{\hat{\beta}} & 0 \end{pmatrix} . \quad (\text{A.28})$$

Since  $\mathfrak{C}$  is symmetric, it holds that

$$\mathfrak{C}_{\hat{\alpha}}^{\hat{\beta}} = \mathfrak{C}^{\hat{\beta}}_{\hat{\alpha}} , \quad (\mathfrak{C}^{-1})^{\hat{\alpha}}_{\hat{\beta}} = (\mathfrak{C}^{-1})_{\hat{\beta}}^{\hat{\alpha}} . \quad (\text{A.29})$$

Given a Dirac spinor  $\Psi$ , its charge conjugate spinor defined by

$$\Psi_{\mathfrak{C}} := \mathfrak{C} \overline{\Psi}^{\text{T}} \quad (\text{A.30})$$

transforms as a Dirac spinor,

$$\delta \Psi_{\mathfrak{C}} = \frac{1}{2} \omega^{\hat{a}\hat{b}} \mathfrak{J}_{\hat{a}\hat{b}} \Psi_{\mathfrak{C}} . \quad (\text{A.31})$$

The transformation laws of  $\Psi$ ,  $\overline{\Psi}$  and  $\Psi_{\mathfrak{C}}$  show that

$$\phi^{\hat{\alpha}} := (\mathfrak{C}^{-1})^{\hat{\alpha}}_{\hat{\beta}} \phi^{\hat{\beta}} = \phi^{\hat{\beta}} (\mathfrak{C}^{-1})_{\hat{\beta}}^{\hat{\alpha}} \quad (\text{A.32})$$

transforms as a dual twistor, while

$$\bar{\phi}_{\hat{\alpha}} := \mathfrak{C}_{\hat{\alpha}}^{\hat{\beta}} \bar{\phi}_{\hat{\beta}} = \bar{\phi}_{\hat{\beta}} \mathfrak{C}^{\hat{\beta}}_{\hat{\alpha}} \quad (\text{A.33})$$

transforms as a twistor. This means that the matrices (A.29) are invariant tensors of  $\text{SU}(2, 2)$  which can be used to convert all underlined twistor indices,  $\underline{\hat{\alpha}}, \underline{\hat{\beta}}, \dots$ , into twistor indices,  $\hat{\alpha}, \hat{\beta}, \dots$ , and therefore to completely get rid of the former.

The sigma matrices with twistor indices are

$$(\Sigma_{\hat{a}})_{\hat{\alpha}\hat{\beta}} := (\Sigma_{\hat{a}})_{\hat{\alpha}\underline{\hat{\gamma}}} \mathfrak{C}^{\underline{\hat{\gamma}}}_{\hat{\beta}} , \quad (\tilde{\Sigma}_{\hat{a}})^{\hat{\alpha}\hat{\beta}} := (\mathfrak{C}^{-1})^{\hat{\alpha}}_{\underline{\hat{\gamma}}} (\Sigma_{\hat{a}})^{\underline{\hat{\gamma}}\hat{\beta}} . \quad (\text{A.34})$$

Since  $(\Gamma_{\hat{a}} \mathfrak{C})^T = -\Gamma_{\hat{a}} \mathfrak{C}$ , these matrices are antisymmetric,

$$(\Sigma_{\hat{a}})_{\hat{\alpha}\hat{\beta}} = -(\Sigma_{\hat{a}})_{\hat{\beta}\hat{\alpha}} , \quad (\tilde{\Sigma}_{\hat{a}})^{\hat{\alpha}\hat{\beta}} = -(\tilde{\Sigma}_{\hat{a}})^{\hat{\beta}\hat{\alpha}} . \quad (\text{A.35})$$

The matrices  $(\Sigma_{\hat{a}})_{\hat{\alpha}\hat{\beta}}$  and  $(\tilde{\Sigma}_{\hat{a}})^{\hat{\alpha}\hat{\beta}}$  obey the relations (A.6). The following completeness relations hold:

$$\frac{1}{2} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} (\tilde{\Sigma}^{\hat{a}})^{\hat{\gamma}\hat{\delta}} = (\Sigma_{\hat{a}})_{\hat{\alpha}\hat{\beta}} , \quad (\text{A.36a})$$

$$\frac{1}{2} \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} (\Sigma_{\hat{a}})_{\hat{\gamma}\hat{\delta}} = (\tilde{\Sigma}^{\hat{a}})^{\hat{\alpha}\hat{\beta}} , \quad (\text{A.36b})$$

$$\frac{1}{2} (\Sigma^{\hat{a}})_{\hat{\alpha}\hat{\beta}} (\Sigma_{\hat{a}})_{\hat{\gamma}\hat{\delta}} = \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} . \quad (\text{A.36c})$$

There is a one-to-one correspondence between complex vectors in  $4+2$  dimensions  $V^{\hat{a}}$  and bi-twistors  $V_{\hat{\alpha}\hat{\beta}} = -V_{\hat{\beta}\hat{\alpha}}$  or  $V^{\hat{\alpha}\hat{\beta}} = \frac{1}{2} \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} V_{\hat{\gamma}\hat{\delta}}$ . It is described by the relations

$$V_{\hat{\alpha}\hat{\beta}} = V^{\hat{c}} (\Sigma_{\hat{c}})_{\hat{\alpha}\hat{\beta}} , \quad V^{\hat{a}} = \frac{1}{4} (\tilde{\Sigma}^{\hat{a}})^{\hat{\gamma}\hat{\delta}} V_{\hat{\gamma}\hat{\delta}} , \quad (\text{A.37a})$$

$$V^{\hat{\alpha}\hat{\beta}} = V^{\hat{c}} (\tilde{\Sigma}^{\hat{c}})^{\hat{\alpha}\hat{\beta}} , \quad V^{\hat{a}} = \frac{1}{4} (\Sigma_{\hat{a}})^{\hat{\gamma}\hat{\delta}} V^{\hat{\gamma}\hat{\delta}} . \quad (\text{A.37b})$$

It is convenient to use the matrix notation  $V := (V_{\hat{\alpha}\hat{\beta}})$  and  $\tilde{V} := (V^{\hat{\alpha}\hat{\beta}})$ . If  $\bar{V}^{\hat{a}}$  is the complex conjugate of  $V^{\hat{a}}$ , then the corresponding bi-twistor matrices  $\tilde{\bar{V}} := (\bar{V}^{\hat{\alpha}\hat{\beta}})$  and  $\bar{V} := (V_{\hat{\alpha}\hat{\beta}})$  are related to each other as in (2.21). Finally, if  $V^{\hat{a}}$  is real, then  $\tilde{\bar{V}} = \tilde{V}$ .

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